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THE AXISYMMETRIC BUCKLING OF INITIALLY IMPERFECT COMPLETE SPHERICAL SHELLS*

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Abstract—Starting out from recent experimental results according to which thin-walled spherical shells subjected to a uniform external pressure deform in an axisymmetric mode at the beginning of the buckling process, the buckling and the postbuckling behavior of complete spherical shells were investigated under the assumption that both the unintentional, random initial deviations from the exact shape, and the following elastic deformations are symmetric to some radius of the shell. The shell was imagined to be decomposed into a cap which shows large deformations, and into a remainder in which the displacements are so small that in their analysis a linear theory can be used satisfactorily. The total potential energy of the system was minimized numerically and in the minimization process the conditions at the boundary between cap and remainder were enforced rigorously. Complete and continuous postbuckling curves were obtained in a number of cases and the maximal value of the pressure parameter was determined in a sufficiently broad range of the geometric parameter to draw conclusions regarding the practical behavior of spherical shells under external pressure. Comparison with earlier theoretical and experimental work yielded satisfactory agreement.

NOTATION

$A_1, A_3, \ldots$	coefficients of stress function defined in equations (27) and (28)
D	bending stiffness $= Eh^3/12(1-\nu^2)$
E	Young's modulus
F	stress function
H, V	horizontal and vertical components of stress resultants
K_{ij}	stiffness coefficients $(i, j = 1, 2)$
M_φ	bending moment resultant in meridional direction
N, T, m, n	integers
P	total potential energy
U	strain energy
V_p	potential of external pressure
$W_{Mr}, W_{Hr}, W_{Vr}, W_{pr}$	work done by M_φ, H and V acting upon the edge of the remainder and by the external pressure acting on the remainder
a	radius of the spherical shell
$b_1, b_3, \ldots$	coefficients of series for β_a^*
h	thickness of shell
k	parameter defined in equation (31)
p	external pressure
p_{cl}	classical buckling pressure
u, w	horizontal and vertical displacements
v	volume displaced during deformation
α	edge angle of the cap
β	change of angle of meridian tangent
$\delta_1, \delta_3, \ldots$	coefficient of series for β_i^*
δ	imperfection amplitude parameter defined in equation (37)

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$\varepsilon_\varphi, \varepsilon_\theta$	midsurface strains in meridional and circumferential directions
$\kappa_\varphi, \kappa_\theta$	midsurface curvature changes in meridional and circumferential directions
λ	geometric parameter $= [12(1-\nu^2)]^{\frac{1}{4}}(a/h)^{\frac{1}{2}}\alpha$
ν	Poisson's ratio
ξ	nondimensional meridional coordinate
ρ	nondimensional pressure parameter
φ	meridional coordinate

Subscripts and superscripts

e	quantities pertaining to the edge of the shell
r	quantities pertaining to the remainder
a	geometric quantities measured from the initial shape of the midsurface of the imperfect shell
i	initial deviations
1, 2	quantities belonging to the primary and secondary states of stress and deformation
$(\ )^*$	nondimensionalization in accordance with equations (7) and (20)
$(\ )'$	differentiation with respect to φ
$(\ )^{\cdot}$	differentiation with respect to ξ

INTRODUCTION

SINCE the pioneering work of Th. von Kármán and H. S. Tsien [1] who attempted to explain the discrepancy between the experimentally obtained buckling pressure and the theoretical classical pressure

$$p_{cl} = (2/[3(1-\nu^2)]^{\frac{1}{2}})(Eh^2/a^2) \tag{1}$$

derived by R. Zoelly [2], E. Schwerin [3] and A. van der Neut [4], the emphasis in the literature has been largely on the solution of the problem of the buckling of clamped shallow spherical shells. The methods used in the case of symmetric buckling can be classified as belonging to one of the following five types: (1) Perturbation method [5, 6]; (2) Power series method [7–11]; (3) Finite difference method [12–14]; (4) Method of iterative numerical integration [15–19]; and (5) Energy method [20]. In the case of nonsymmetric buckling, the bifurcation point of the axisymmetric equilibrium curve at which anti-symmetric equilibrium states become possible for the first time was sought in agreement with Koiter's theory [21–25]. Accurate solutions of the axisymmetric buckling of clamped shallow spherical shells were obtained by H. J. Weinitschke [11], B. Budiansky [15], G. A. Thurston [17], R. R. Archer [14] and D. Bushnell [20]. The most accurate results relating to nonsymmetric buckling are probably those obtained by N. C. Huang [24] and H. J. Weinitschke [25]. The values of the buckling pressure are usually plotted against the geometric parameter λ defined as

$$\lambda = [12(1-\nu^2)]^{\frac{1}{4}}(a/h)^{\frac{1}{2}}\alpha \tag{2}$$

where α is the edge-angle of the shallow spherical shell measured from the axis of symmetry.

In 1961, A. G. Gabriliants and V. I. Feodosev [26] treated the problem of the buckling of the complete and perfect spherical shell by dividing it into a shallow spherical cap and a remainder. They solved the nonlinear differential equations of the shallow spherical shell with boundary conditions derived from the solution of the linear differential equations governing the bending of the remainder. The numerical calculations were performed by a finite difference method and only the minimal value of the postbuckling equilibrium pressure was sought.

In 1962, J. M. T. Thompson [27] presented the results of an experimental and theoretical investigation. The experiments were performed on polyvinyl chloride shells of a radius-to-thickness ratio of about 20 with a loading device which was extremely rigid. In the theoretical work, the complete spherical shell was divided into a shallow spherical cap and a remainder. The bending deformations were assumed to be confined to the cap, and the remainder was assumed to undergo only a membrane contraction. Thus, the matching of the cap and the remainder was accomplished by enforcing the conditions that the meridional displacements of the cap and the rotation of the meridian should vanish at the boundary of the two regions. The calculations were carried out with the aid of the Rayleigh–Ritz method and the normal displacement was assumed in the form of a finite polynomial with three undetermined coefficients. The total potential energy was minimized with respect to four free parameters, namely the three undetermined coefficients of the polynomial assumed and the parameter representing the edge angle of the cap. The effect of initial imperfections was also studied for a particular shape of the imperfection. Good agreement was observed between the theoretical results and the experiment.

In 1964, A. B. Sabir [28] investigated the interaction between a uniform external pressure and a point load applied to a spherical shell. He assumed that the dimple formed by the application of the point load has a shallow spherical shape with the initial curvature completely reversed. He stipulated that the bending deformations were confined to edge zones, and the edge zones were very narrow both for the dimple and the remainder. Thus, the essentially nonlinear problem of large deformations was treated by the linear bending theory of shallow spherical shells. The boundary between dimple and remainder was taken as the point at which the tangent to the meridional section is perpendicular to the axis of rotation of the shell. Assuming that the imperfection has the same shape as the dimple formed by the point load, Sabir reinterpreted the effect of the point load as the effect of the initial imperfection.

In 1964, J. M. T. Thompson [29] applied Koiter's ideas to an investigation of the initial, axisymmetric postbuckling behavior of a complete spherical shell by using the strain-displacement relations valid for shallow spherical shells. A closed-form expression was derived for the initial slope of the postbuckling path. This slope was found to be negative indicating sensitivity to initial imperfections. The Koiter approach was also employed by J. W. Hutchinson [30]. His analysis, published in 1967, was not restricted to axisymmetric buckling, but took into account the coupling of all the buckling modes associated with the bifurcation point.

More details of the historical developments can be found in a survey article written by Th. von Kármán and A. D. Kerr [31].

A series of experiments on the buckling of complete spherical shells was recently performed in the Department of Aeronautics and Astronautics of Stanford University. The test results obtained by R. L. Carlson, R. L. Sendelbeck and N. J. Hoff [32] and by L. Berke [33] seem to indicate that the buckling shape is axisymmetric, at least in the important early phases of the buckling process. In the present study, use is made of these experimental observations. It is assumed that the shell has an axisymmetric initial imperfection. In the analysis, the complete spherical shell is divided into two parts; the shallow spherical cap in which the initial imperfection exists and the dimple forms, and the remaining portion of the shell most of which deforms in simple contraction. It is further assumed that the deformation in the cap is large. Hence the nonlinear theory of finite deformation is used for the analysis of the cap. The bending deformations of the remainder occur in an edge zone and are assumed to be small. Hence the linear bending theory is used for the analysis of the remainder, but the effect of the prestress of the fundamental state is duly considered.

After the major portion of the work here described was completed, the authors became aware of four additional papers whose topics are related to that of the present report. In the first one of these, Thurston and Penning [34] describe tests carried out with forty aluminum alloy spherical caps after their exact shape had been carefully measured. The authors also developed a method of numerical integration of the large-displacement Reissner equations which permitted them to calculate the changes in the shape of each specimen as the external pressure was increased. Comparison of calculation and experiment proved the correctness of the basic assumptions and of the method of analysis although the maximum of the pressure predicted by theory often differed considerably from the collapse pressure measured, particularly when the a/h ratio of the shell was large.

In the second paper Bushnell [35] used a numerical integration scheme for the solution of finite difference equations to determine the collapse pressure of hemispheres with a flat spot. His calculations were also based on Reissner's equations and only axisymmetric deformations were taken into account. In a follow-up paper, Bushnell [36] used the digital computer to determine first the changes in the axisymmetric mode of prebuckling deformations of spherical shells with a flat spot, and then the possibility of a bifurcation of the equilibrium into a multilobed state of deformations. At a distance of $3{\cdot}3(ah)^{\frac{1}{2}}$ from the edge of the flat spot the radial displacement and the slope of the displacements accompanying buckling were taken as zero. This was assumed to represent a satisfactory approximation to the behavior of complete spherical shells. Further reference to Bushnell's work will be made in the body of the paper.

In a fourth paper Krenzke and Kiernan [37] reported on tests carried out at the David Taylor Model Basin with sixty-two aluminum alloy shells machined very accurately out of bar stock. Thirty-six of these had predetermined imperfections in the form of a flat spot.

BASIC EQUATIONS

The basic equations governing the axisymmetric large deformations but small strains of rotationally symmetric shells were formulated by E. Reissner [38]. They can be simplified to hold for shallow spherical shells with finite displacements and small rotations of the meridian. If there exists a small initial imperfection characterized by the initial angle β_i between the tangents to the meridians of the imperfect and perfect shells and if the geometric quantities measured from the midsurface of the initially imperfect spherical shell are designated by the subscript a (for additional), the strain–displacement relations and the compatibility equation become (see Figs. 1–3):

(a) *Strain–displacement relations*

$$\begin{aligned}\varepsilon_{\varphi a} &= (u_a'/a)+\varphi\beta_a+(\beta_a^2/2)+\beta_i\beta_a,\\ \varepsilon_{\theta a} &= u_a/a\varphi,\\ \kappa_{\varphi a} &= -\beta_a'/a,\\ \kappa_{\theta a} &= -\beta_a/a\varphi,\end{aligned} \tag{3}$$

where u is the horizontal component of the displacement.

TOTAL POTENTIAL ENERGY

The total potential energy of the complete spherical shell can be separated into the following parts:

(a) Strain energy stored in the cap;
(b) Potential of the external pressure acting upon the cap; and
(c) Total potential energy of the remainder.

(a) *Strain energy stored in the cap, U*

U can be given as the sum of the strain energy due to membrane stresses and the strain energy due to bending. It can be written in terms of F and β_a as

$$U = (\pi/Eh)\int_0^\alpha [(F/\varphi)^2+(F')^2-(2\nu FF'/\varphi)]\varphi\,\mathrm{d}\varphi+(\pi a^2 D)\int_0^\alpha [(\beta_a'/a)^2 + (\beta_a/a\varphi)^2+(2\nu\beta_a\beta_a'/a^2\varphi)]\varphi\,\mathrm{d}\varphi, \tag{9}$$

where $D = Eh^3/12(1-\nu^2)$ is the bending rigidity of the wall of the shell.

(b) *Potential of the external pressure acting upon the cap, V_p*

V_p is given in the shallow-shell approximation by

$$V_p = -2\pi a^2 p\int_0^\alpha \varphi w_a\,\mathrm{d}\varphi, \tag{10}$$

where w_a is the vertical component of the displacement measured from the midsurface of the initially imperfect shell and p is the external pressure.

Integrating equation (10) by parts, one obtains

$$V_p = -\pi(a\alpha)^2 p w_a(\alpha)+\pi a^2 p\int_0^\alpha \varphi^2 w_a'\,\mathrm{d}\varphi\,. \tag{11}$$

As it will be shown under (c), it is permissible to omit the first term in the right-hand member of equation (11). In other words, one is allowed to take

$$w_a(\alpha) = 0 \tag{12}$$

without loss of generality, if the work done by the vertical component of the stress resultant acting upon the edge of the remainder is also omitted. Thus one has

$$V_p = \pi a^3 p\int_0^\alpha \varphi^2\beta_a\,\mathrm{d}\varphi \tag{13}$$

where the relation

$$\beta_a = (1/a)(\mathrm{d}w_a/\mathrm{d}\varphi)$$

has been used.

(c) *Total potential energy of the remainder, P_r*

The quantities characterizing the deformations of the remainder will be represented as the result of the superposition of a primary state and secondary state.

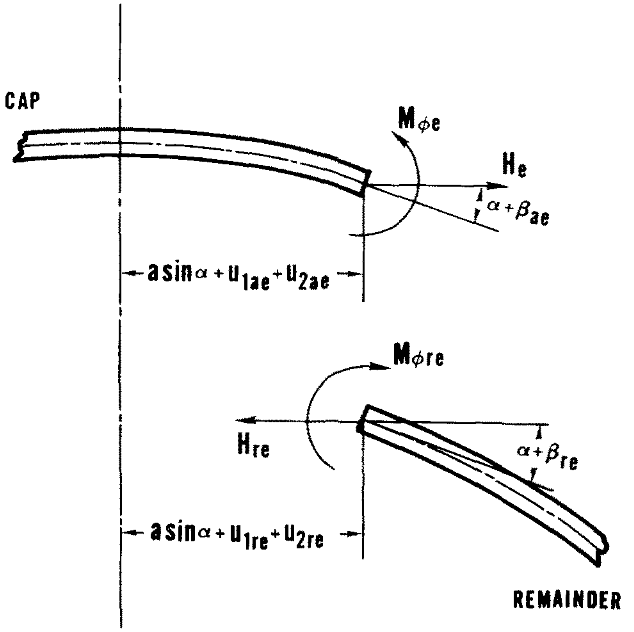

FIG. 3. Sign convention for edge loads and edge displacements.

(b) *Compatibility equation*

$$\varepsilon_{\theta a}+\varphi\varepsilon'_{\theta a}-\varepsilon_{\varphi a} = -\varphi\beta_a-(\beta_a^2/2)-\beta_i\beta_a, \tag{4}$$

where the prime indicates differentiation with respect to φ, and a is the radius of the spherical shell.

The stress function F is defined by the equations

$$F = a\varphi N_\varphi, \qquad F' = aN_\theta. \tag{5}$$

Then the quantities $\varepsilon_{\varphi a}$, $\varepsilon_{\theta a}$ and u_a can be written in terms of F:

$$\begin{aligned}\varepsilon_{\varphi a} &= (1/Eha)[(F/\varphi)-\nu F'],\\ \varepsilon_{\theta a} &= (1/Eha)[F'-(\nu F/\varphi)],\\ u_a &= (1/Eh)(\varphi F'-\nu F),\end{aligned} \tag{6}$$

where E, ν and h are Young's modulus, Poisson's ratio and the wall-thickness of the shell, respectively.

The compatibility equation (4) can now be written in terms of F and β_a. If the dimensional quantities are nondimensionalized by introduction of the relations

$$\xi = \varphi/\alpha, \qquad \beta^* = \beta/\alpha, \qquad F^* = F/Eha\alpha^3, \tag{7}$$

then the compatibility equation can be written in the nondimensional form

$$F^{*\cdot\cdot}+(F^{*\cdot}/\xi)-(F^*/\xi^2) = -\beta_a^*-(\beta_a^{*2}/2\xi)-(\beta_i^*\beta_a^*/\xi), \tag{8}$$

where the dot indicates differentiation with respect to ξ.

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has been used.

(c) *Total potential energy of the remainder, P_r*

The quantities characterizing the deformations of the remainder will be represented as the result of the superposition of a primary state and secondary state.

The primary state of stress and deformation, indicated by the subscript 1, is defined as the membrane state of stress and deformation of the perfect shell. It can be represented entirely by the normal stress resultants $N_{\varphi 1}$ and $N_{\theta 1}$ and by the horizontal displacement u_1. The secondary state of stress and deformation, indicated by the subscript 2, is defined as the state of stress and deformation produced by the addition of edge force and moment resultants to the state of loading of the primary state. Since $M_{\varphi 1}$, $M_{\theta 1}$, β_1, $\kappa_{\theta 1}$ vanish identically in the present approximation, M_φ, M_θ, β, κ_φ and κ_θ are quantities appearing in the secondary state only. Thus these quantities will be written without the subscript 2.

In addition to the subscripts 1 and 2, the subscripts e and r are introduced to characterize the quantities pertaining to the edge and to the remainder. No subscript is used to denote the quantities pertaining to the cap, except a for the displacements and rotation of the meridian of the imperfect shell.

The strain energy U_r stored in the remainder when it distorts in the secondary state under the action of the stress resultants acting upon it from the cap is

$$U_r = W_{Mr} + W_{Hr} + W_{Vr} + W_{pr}, \tag{14}$$

where W_{Mr}, W_{Hr}, W_{Vr} and W_{pr} are the work done by the stress resultants $M_{\varphi re}$, H_{re} and V_{re} and by the external pressure p, respectively. The distortions are assumed to begin after the remainder had been compressed uniformly by the pressure p; hence W_{pr} is the work done by the constant pressure p during the deformations caused by $M_{\varphi re}$, H_{re} and V_{re}. Since the total potential energy P_r of the remainder is

$$P_r = U_r + V_{pr},$$

and because

$$V_{pr} = -W_{pr},$$

when the external pressure is constant, one can write

$$P_r = U_r - W_{pr} = W_{Mr} + W_{Hr} + W_{Vr}. \tag{15}$$

Two terms are still missing from this equation, namely the strain energy stored in the remainder during the deformations corresponding to the uniform compression of the primary state, and the corresponding potential of the external pressure. But these quantities are independent of the values of u_{2re} and β_{re} and thus they do not make any contributions when the total potential energy of the entire system is varied. It is permissible, therefore, to omit these terms from this analysis.

The first term in the right-hand member of equation (11) can be written in the form

$$\{-\pi(a\alpha)^2 p w_{1ae} - \pi(a\alpha)^2 p w_{2ae}\}. \tag{16}$$

The second term of this expression is the same, except for the sign, as the last term in the right-hand member of equation (15). Since the potential energy expressions for the cap and remainder must be added to obtain the total potential energy of the entire spherical shell, the term W_{Vr} cancels the second term in the expression (16). The remaining term of (16), $-\pi(a\alpha)^2 p w_{1ae}$, is independent of u_{2re} and β_{re} and thus, for the reason stated previously, can be omitted. This justifies the assumption of equation (12). One may, therefore, calculate the total potential energy of the remainder from the equation

$$\begin{aligned} P_r &= W_{Mr} + W_{Hr} \\ &= \pi a\alpha[u_{1re}H_{1re} + u_{1re}H_{re} - H_{1re}u_{re} - u_{re}H_{re} + \beta_{re}M_{\varphi re}] \end{aligned} \tag{17}$$

where the subscript $1re$ indicates a quantity related to the edge of the remainder in the primary state, and re indicates a total displacement or force quantity (sum of primary and secondary) for the edge of the remainder.

The assumption of small deformations for the remainder enables one to make use of the well-established solutions of the linear equations governing the deformations of the pressurized spherical shell. According to the solutions of the linear theory, H_{re} and $M_{\varphi re}$ are related linearly to u_{re} and β_{re}. The relations are written in the form

$$\begin{aligned} H_{re} &= K_{11r}(u_{re}-u_{1re})+K_{12r}\beta_{re}+H_{1re}, \\ M_{\varphi re} &= K_{21r}(u_{re}-u_{1re})+K_{22r}\beta_{re}, \end{aligned} \tag{18}$$

where K_{ijr} $(i, j = 1, 2)$ are the stiffness coefficients for the remainder, which are presented in the Appendix. The sign convention for the edge loads and edge displacements is shown in Fig. 3.

MATCHING CONDITIONS

Since the cap and remainder are parts of the complete spherical shell, there should not exist any discontinuity in the physical quantities characterizing the state of the shell at $\varphi = \alpha$. Therefore, the following four conditions are imposed to ensure the proper matching of the cap and remainder:

$$\begin{aligned} u_{ae} &= u_{re}, \qquad \beta_{ae} = \beta_{re}, \\ H_e &= H_{re}, \qquad M_{\varphi e} = M_{\varphi re}. \end{aligned} \tag{19}$$

The first two conditions can be satisfied simply by substitutions. The remaining two equations can be reduced to two equations in F_e and β_{ae} with the aid of equations (6) and (18). The dimensional quantities are nondimensionalized through introduction of the following relations:

$$\begin{aligned} K_{11}^* &= (a\lambda\alpha/Eh)K_{11}, \\ K_{12}^* &= (\lambda^2/Eh\alpha)K_{12}, \\ K_{22}^* &= (\lambda^3/a\alpha^3 Eh)K_{22}, \\ K_{21}^* &= -K_{12}^*, \\ \rho &= p/p_{cl}. \end{aligned} \tag{20}$$

Then the two equations become in nondimensional form:

$$(K_{11r}^*/\lambda)F_e^{*\cdot}-(\nu K_{11r}^*/\lambda)F_e^*-F_e^*+(K_{12r}^*/\lambda^2)\beta_{ae}^*+(2\rho/\lambda^2)[(1-\nu)(K_{11r}^*/\lambda)-1] = 0, \tag{21}$$

$$\lambda^2 K_{12r}^*(F_e^{*\cdot}-\nu F_e^*)-\beta_{ae}^{*\cdot}-\nu\beta_{ae}^*-\lambda K_{22r}^*\beta_{ae}^*+2\rho(1-\nu)K_{12r}^* = 0. \tag{22}$$

The total potential energy can now be written in terms of F and β_a only. The result is in nondimensional form

$$\begin{aligned} P^* = {} & (\lambda^2/2)\int_0^1 [(F^*/\xi)^2+(F^{*\cdot})^2-2\nu(F^*F^{*\cdot}/\xi)]\xi\,d\xi \\ & +(1/2\lambda^2)\int_0^1 [(\beta_a^*/\xi)^2+(\beta_a^{*\cdot})^2+2\nu(\beta_a^*\beta_a^{*\cdot}/\xi)]\xi\,d\xi+2\rho\int_0^1 \beta_a^*\xi^2\,d\xi \\ & +[\rho(F_e^{*\cdot}-F_e^*)-(\lambda^2/2)(F_e^{*\cdot}-\nu F_e^*)F_e^*-(1/2\lambda^2)(\beta_{ae}^{*\cdot}+\nu\beta_{ae}^*)\beta_{ae}^*]. \end{aligned} \tag{23}$$

METHOD OF SOLUTION

The Rayleigh–Ritz procedure is used to obtain the approximate solution of the non-linear problem. β_a^* is assumed in the form of a finite polynomial in ξ with undetermined coefficients. It contains only odd powers of ξ because the deformations are axially symmetric:

$$\beta_a^* = \sum_{n=1,3,5}^{N} b_n \xi^n, \tag{24}$$

where N is an odd integer and b_n are the undetermined coefficients.

The initial imperfection is also assumed to be given in the form of a finite polynomial

$$\beta_i^* = \sum_{n=1,3}^{N} \delta_n \xi^n \tag{25}$$

where the δ_n are prescribed for a particular imperfection shape. In practice, some of the δ_n will be chosen as zero, but having the upper limit N the same in the sums of equations (24) and (25) is convenient when programming for the computer.

β_a^* and β_i^*, as assumed in the form of equations (24) and (25), are substituted into the right-hand member of the compatibility equation, equation (8). Then equation (8) can be easily integrated. The result can be written in the form

$$F^* = \sum_{n=1}^{2N+1} A_n \xi^n, \tag{26}$$

where $A_3, A_5, \ldots$ are defined as

$$\begin{aligned} A_3 &= (\tfrac{1}{8})[(1+\delta_1)b_1 + (\tfrac{1}{2})b_1^2] \\ A_5 &= -(\tfrac{1}{24})[(1+\delta_1)b_3 + \delta_3 b_1 + b_1 b_3] \\ A_7 &= -(\tfrac{1}{48})[(1+\delta_1)b_5 + \delta_3 b_3 + \delta_5 b_1 + (\tfrac{1}{2})b_3^2 + b_1 b_5] \ldots, \end{aligned} \tag{27}$$

and A_1 is the integration constant which will be determined with the aid of one of the matching conditions.

Substituting F^* and β_a^* into equations (21) and (22), one obtains the matching conditions written in terms of b_n and A_n. The result is

$$\sum_{n=1}^{2N+1} [(n-\nu)(K_{11r}^*/\lambda)-1]A_n + (K_{12r}^*/\lambda^2)\sum_{n=1}^{N} b_n + (2\rho/\lambda^2)[(1-\nu)(K_{11r}^*/\lambda)-1] = 0, \tag{28}$$

$$\sum_{n=1}^{2N+1} \lambda^2 K_{12r}^*(n-\nu)A_n - \sum_{n=1}^{N} (\lambda K_{22r}^* + n + \nu)b_n + 2\rho(1-\nu)K_{12r}^* = 0. \tag{29}$$

From one of these matching conditions, say equation (28), the integration constant A_1 is determined. One has

$$A_1 = -\sum_{n=3}^{2N+1} (1/k)[(n-\nu)(K_{11r}^*/\lambda)-1]A_n - (1/k)(K_{12r}^*/\lambda^2)\sum_{n=1}^{N} b_n - (2\rho/\lambda^2) \tag{30}$$

where

$$k = (1-\lambda)(K_{11r}^*/\lambda) - 1. \tag{31}$$

Substituting F^* and β_a^* into equation (23), one obtains the total potential energy written in terms of b_n and A_n:

$$P^* = (\lambda^2/2)\sum_{n=1}^{2N+1}\sum_{m=1}^{2N+1}\left[\frac{(1-n^2)+\nu(m-n)}{n+m}\right]A_nA_m$$

$$+(1/2\lambda^2)\sum_{n=1}^{N}\sum_{m=1}^{N}\left[\frac{(1-n^2)+\nu(n-m)}{n+m}\right]b_nb_m$$

$$+\sum_{n=1}^{2N+1}\rho(n-1)A_n+2\rho\sum_{n=1}^{N}b_n/(n+3). \tag{32}$$

For given values of ρ, λ and β_i^* the total potential energy P^* must be a minimum with respect to all b_n in the presence of the constraining condition, equation (29). The minimization is performed with the aid of a Lagrange multiplier; it yields a set of the nonlinear algebraic equations in b_n, which is solved with the aid of the Newton–Raphson iterative procedure.

Once the b_n are obtained, the A_n are calculated according to equations (27) and (30). Then the basic quantities β_a^*, u_a^*, w_a^*, H^* and M_ξ^* can be calculated for the cap.

The volume displaced during deformations is given in nondimensional form as

$$v^* = -2\int_0^1 \beta_a^*\xi^2\,\mathrm{d}\xi. \tag{33}$$

Substitution from equation (24) and integration yield

$$v^* = -2\sum_{n=1}^{N} b_n/(n+3). \tag{34}$$

NUMERICAL RESULTS

A computational program for the minimization and the Newton–Raphson processes, valid for power series of β_a^* and β_i^* of any odd number of terms greater than three, was developed and written in ALGOL machine language. The computations were carried out on the Burroughs B5500 Digital Computer of the Computation Center of Stanford University.

The computations were performed for two types of initial imperfections; one was in the form of a dimple and the other was a spherical region whose radius of curvature was larger than that of the perfect spherical shell. The former is called the imperfection of type 1, and the latter the imperfection of type 2. The approximate shapes of these imperfections are sketched in Figs. 4a and 4b. In this analysis β_i^* is taken in the form:

For the imperfection of type 1:

$$\beta_i^* = -6[12(1-\nu^2)]^{\frac{1}{4}}(1/\lambda^2)\delta\xi(1-\xi^2)^2 \tag{35}$$

and for the imperfection of type 2:

$$\beta_i^* = -2[12(1-\nu^2)]^{\frac{1}{4}}(1/\lambda^2)\delta\xi \tag{36}$$

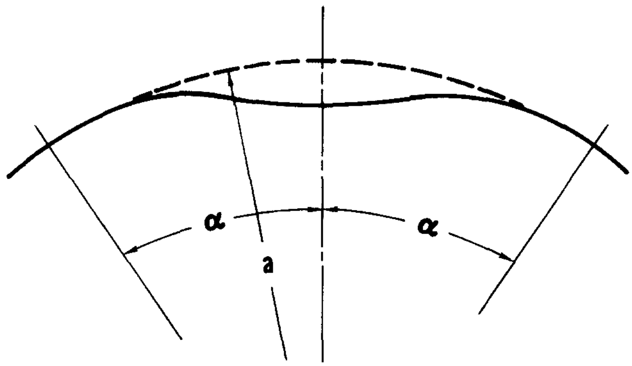

FIG. 4a. Imperfection of type 1.

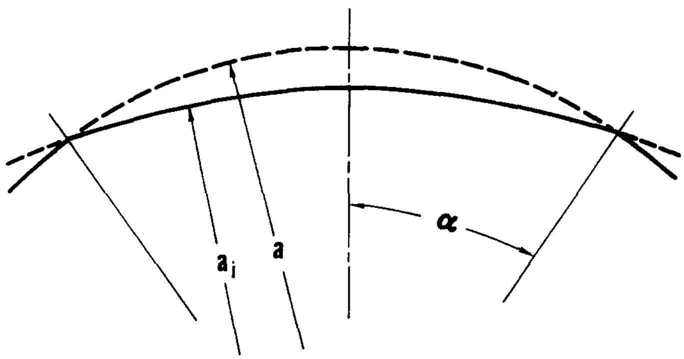

FIG. 4b. Imperfection of type 2.

where δ is the ratio of the initial deviation at the axis of rotation, $w_i(0)$, to the wall-thickness of the shell, h:

$$\delta = w_i(0)/h. \tag{37}$$

The effect of imperfection 1 was studied by Budiansky [15], and that of imperfection 2 by Bushnell [35, 36]. Krenzke and Kiernan [37] carried out tests with specimens having imperfections of type 2.

For each of several pairs of values of λ and δ the values of b_n were guessed for a prescribed value of δ. These values were improved through repeated solution of linear equations in accordance with the Newton–Raphson procedure until the changes in the absolute values of the three most important coefficients were less than one-hundredth of one per cent of the absolute values obtained in the previous iteration. Next the number of coefficients b_n was increased by two, and the procedure outlined was repeated. The volume displaced in consequence of the bending deformations was calculated and a comparison was made between the values obtained for q terms and for $q+2$ terms of the b_n series. When the absolute value of the difference was smaller than one-hundredth of one per cent of the absolute value of the displaced volume obtained for q terms, the values of b_n obtained in the last calculation were considered as the solution of the equilibrium problem for the values of λ, δ and ρ chosen.

Next the values of ρ were increased stepwise and the calculations were repeated. The increment was first taken as 0·1, and when the maximal, or critical, value of ρ was closely approached, the increment in ρ was reduced to 0·01. The critical value of ρ was defined as the largest value of ρ for which the procedure first outlined converged.

The results of the calculations are presented in Figs. 5 and 6 in the form of ρ_{cr} vs. λ

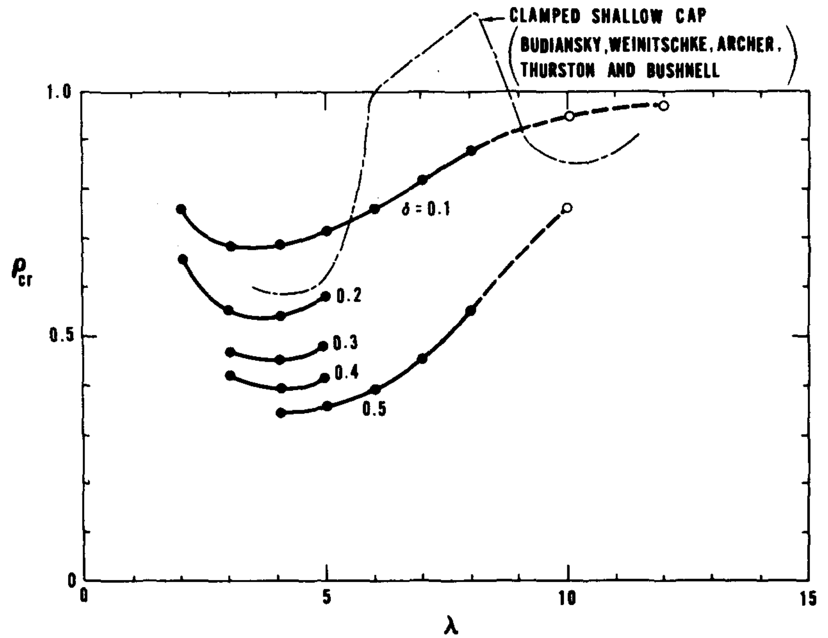

FIG. 5. Critical pressure parameter ρ_{cr} vs. geometric parameter λ; imperfection of type 1.

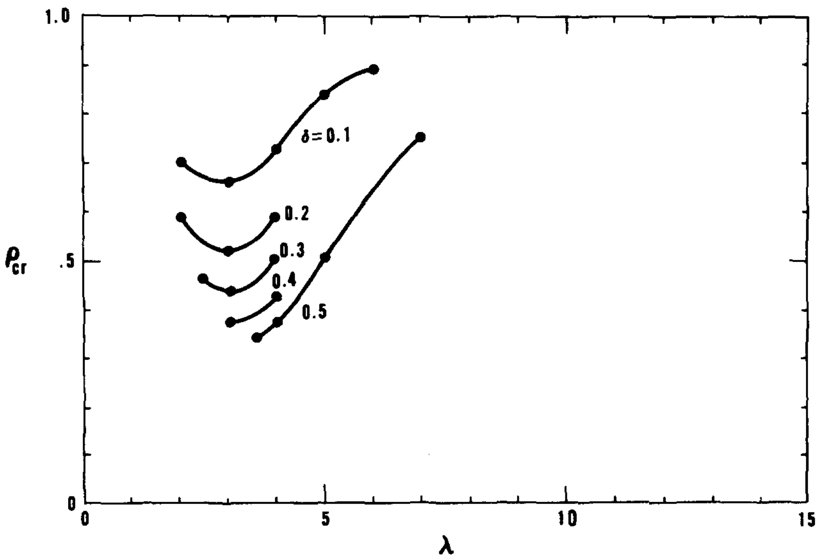

FIG. 6. Critical pressure parameter ρ_{cr} vs. geometric parameter λ; imperfection of type 2.

curves. For large values of λ, serious difficulties in finding convergent solutions were encountered. Values of ρ_{cr} calculated in this range are shown by circles connected with broken lines. It is worth noting that Bushnell [36] found bifurcation into a multilobed pattern at these high values of λ.

From Figs. 5 and 6 one can see that there exists a minimum of ρ_{cr} in each ρ_{cr} vs. λ curve. It occurs at about $\lambda = 4$ for the imperfection of type 1, and at $\lambda = 3$ for the imperfection of type 2. If a/h is given, the angle α, which indicates the angular extent of the initial imperfection, can be calculated for each given value of λ. Thus the most dangerous angular extent of the imperfection can be determined by assigning the values 4 and 3 to λ in equation (2) for the two types of imperfections. For example, if the shell is characterized by $a/h = 10^3$, then the most dangerous angular extent of the imperfection is obtained as

$$\alpha = 4^0 \text{ for the imperfection of type 1}$$
$$\alpha = 3^0 \text{ for the imperfection of type 2}$$

In the case of the particular values $\lambda = 4$ and 3 for the imperfections of type 1 and 2, respectively, additional calculations have also been performed.

In the neighborhood of the maximal value of ρ the convergence of the procedure could be improved by prescribing the volume parameter v^* rather than the pressure parameter ρ. The same approach was used near the minimal values of ρ when the complete equilibrium curves for $\delta = 0{\cdot}1$ were calculated. The results are presented in Figs. 7 and 8. The equilibrium states of prebuckling, unstable postbuckling and stable postbuckling are indicated by the numbers (I), (II) and (III), respectively.

The portions (I) and (II) of the complete equilibrium curves are presented in Figs. 9 and 10 for various values of δ. In these figures the crossed lines indicate the solutions obtained by prescribing the values of the volume parameter v^*. The change in volume due to the change in the fundamental state of uniform compression is not included in the figures.

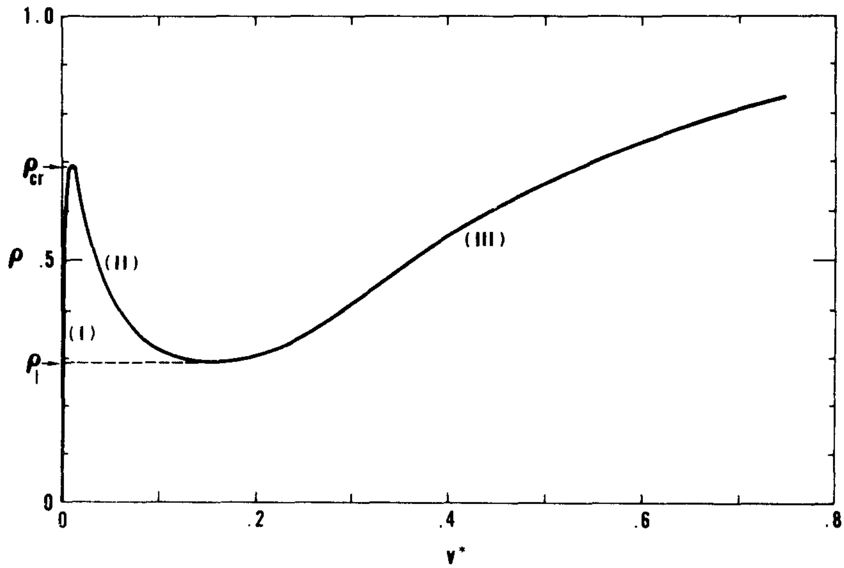

FIG. 7. Complete pressure–volume curve; $\lambda = 4$, $\delta = 0{\cdot}1$, imperfection of type 1 (volume changes caused by changes in the membrane stresses in the remainder are not included).

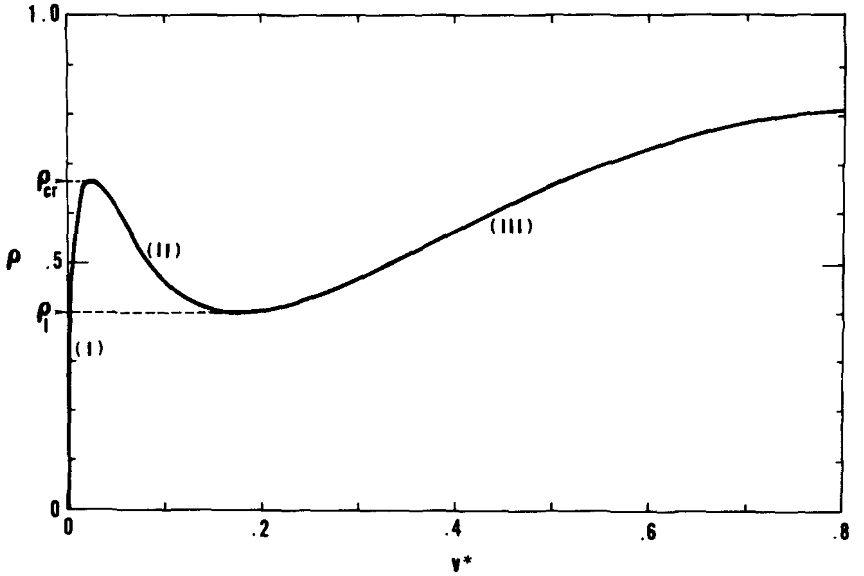

FIG. 8. Complete pressure–volume curve; $\lambda = 3, \delta = 0{\cdot}1$, imperfection of type 2 (volume changes caused by changes in the membrane stresses in the remainder are not included).

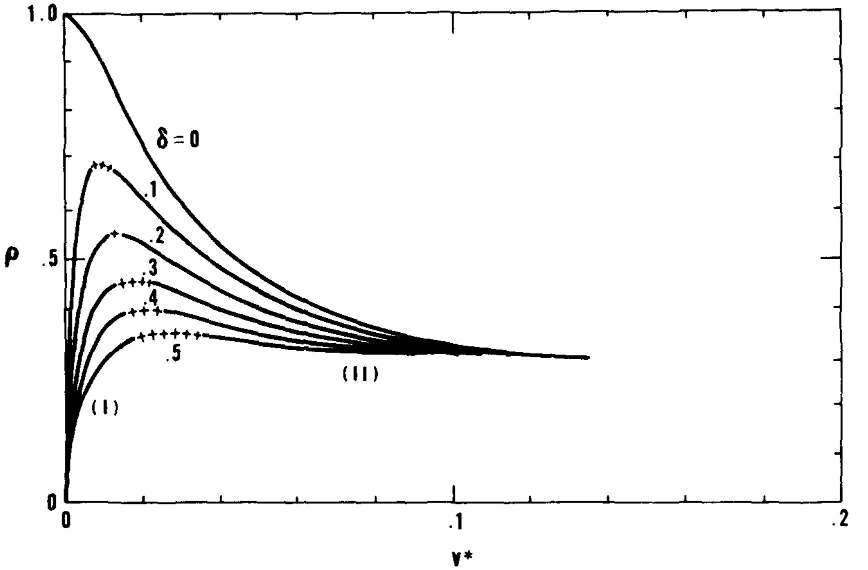

FIG. 9. Early phases of pressure–volume curve; $\lambda = 4$, imperfection of type 1. The crossed line + + + indicates the part of the curve obtained by prescribing v^* in the iteration process (volume changes caused by changes in the membrane stresses in the remainder are not included).

The relation between δ and ρ_{cr} is presented in Fig. 11. This shows that the spherical shell is sensitive to both types of imperfection. The results of the present study are compared with those obtained by A. B. Sabir [28] and J. W. Hutchinson [30]. When the minimal values of ρ_{cr} were cross-plotted from Bushnell's [35] Fig. 7, the points obtained fell on the curve labeled "Type 2" of Fig. 11 of the present report so accurately that no separate curve

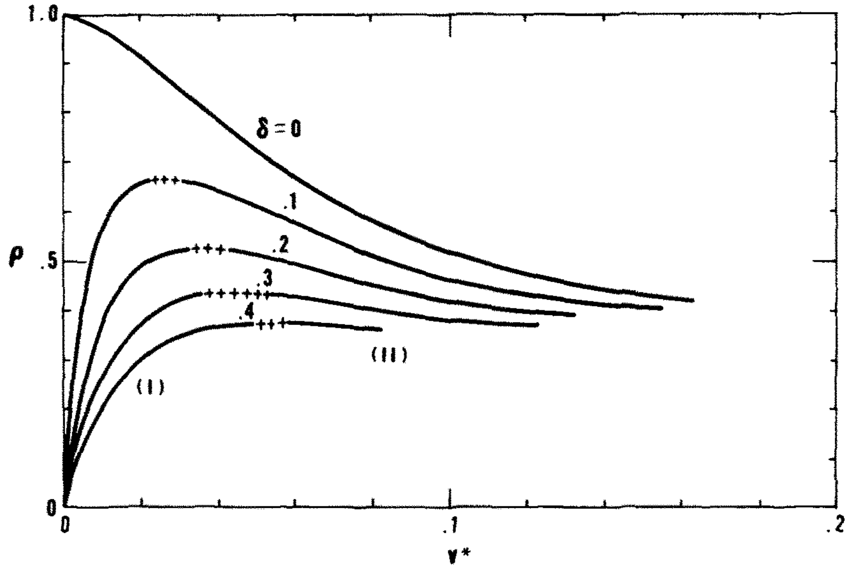

FIG. 10. Early phases of pressure–volume curve; $\lambda = 3$, imperfection of type 2. The crossed line + + + indicates the part of the curve obtained by prescribing v^* in the iteration process (volume changes caused by changes in the membrane stresses in the remainder are not included).

could be drawn. This is a satisfactory check of the accuracy of the calculations because the two solutions were obtained in quite different ways.

A second check was made by comparing the experimental buckling pressures of Krenzke and Kiernan [37] with the ρ_{cr} values shown in Fig. 6. In this comparison only the thinnest shells of the experimental series were included. In the others the elastic limit of the material must have been exceeded and the present theory is valid only for perfectly

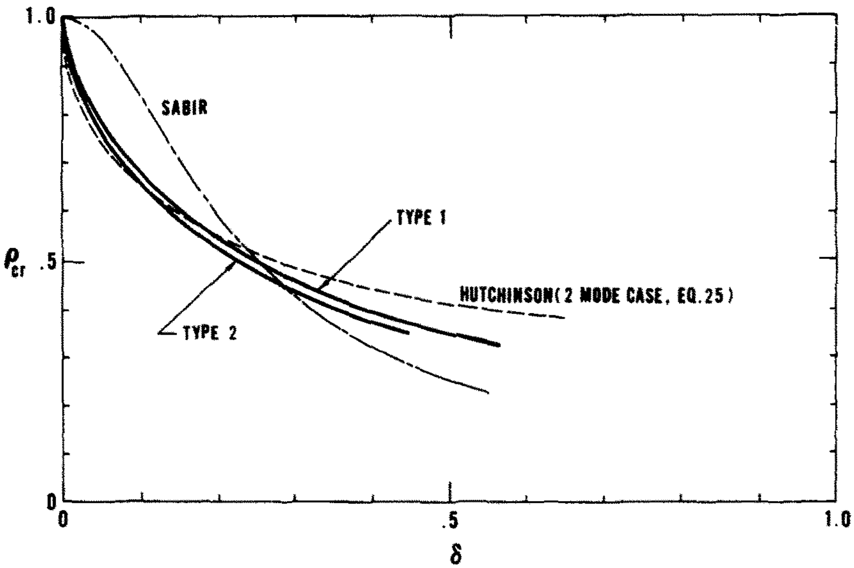

FIG. 11. Critical pressure parameter vs. imperfection amplitude.

elastic shells. Great accuracy cannot be claimed for the comparison because values of ρ_{cr} had to be interpolated, and in some cases even extrapolated, from the curves of Fig. 6. Nevertheless satisfactory agreement was obtained between theory and experiment as the experimental buckling pressures always fell between 69 per cent and 117 per cent of the theoretical values.

CONCLUSION

The maximal values of the pressure parameter ρ in the equilibrium state before buckling have been calculated for many combinations of the geometric parameter λ and the imperfection parameter δ. It can be seen that for all values of δ the lowest buckling pressure is reached when λ is about 4 for the imperfection in the shape of a dimple, and when λ is about 3 for the imperfection in the shape of a spherical region.

In the manufacture of spherical shells for industrial purposes imperfections cannot be avoided. They are distributed in a random manner over the entire surface of the spherical shell. If the imperfections are of the two types that have been investigated in the present study, buckling is most likely to be observed at that point on the spherical shell where the initial imperfection corresponds to $\lambda = 4$ or 3. The maximal value of the pressure that can be reached in a pressurization test is then given by the two almost coincident curves shown in Fig. 11.

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APPENDIX

Influence and stiffness coefficients for internally pressurized spherical shells were obtained by G. Cline [39]. D. Bushnell [40] provided a supplement to Cline's development by investigating the externally pressurized spherical shells. His coefficients can be simplified to hold for shallow spherical shells by approximating $\sin\varphi$ and $\cos\varphi$ by φ and 1, respectively. The simplification yields the following stiffness coefficients:

$$K_{11} = (Eh/a\lambda\alpha)(\Delta^*/\Delta_0^*)\{[1-\rho^2]^{\frac{1}{2}}(R_1^2+I_1^2)\}$$

$$K_{12} = -K_{21} = (\alpha Eh/\lambda^2)(\Delta^*/\Delta_0^*)\{[(1+\rho)/2]^{\frac{1}{2}}(R_0I_1-I_0R_1)-[(1-\rho)/2]^{\frac{1}{2}}(I_0I_1+R_0R_1)\}$$

$$\begin{aligned}K_{22} = &-(a\alpha^3Eh/\lambda^3)(\Delta^*/\Delta_0^*)\{[1-\rho^2]^{\frac{1}{2}}(R_0^2+I_0^2)+(1-\nu^2)(1/\lambda^2)[1-\rho^2]^{\frac{1}{2}}(R_1^2+I_1^2)\\&+(2/\lambda)[1-\rho(1+\nu)][(1+\rho)/2]^{\frac{1}{2}}(R_1I_0-R_0I_1)\\&-(2/\lambda)[1+\rho(1+\nu)][(1-\rho)/2]^{\frac{1}{2}}(R_0R_1+I_0I_1)\}\end{aligned}$$

where

$$\Delta^* = (1+2\rho)[(1-\rho)/2]^{\frac{1}{2}}(R_1R_0+I_1I_0)+(1-2\rho)[(1+\rho)/2]^{\frac{1}{2}}(I_1R_0-R_1I_0)$$
$$+(\nu-1)(1/\lambda)[1-\rho^2]^{\frac{1}{2}}(R_1^2+I_1^2)$$
$$\Delta_0^* = (1-\nu^2)(1/\lambda^2)(1-\rho^2)(R_1^2+I_1^2)^2$$
$$+(2/\lambda)[1-\rho(1+\nu)](1+\rho)[(1-\rho)/2]^{\frac{1}{2}}(R_1I_0-R_0I_1)(R_1^2+I_1^2)$$
$$-(2/\lambda)[1+\rho(1+\nu)](1-\rho)[(1+\rho)/2]^{\frac{1}{2}}(R_1R_0+I_0I_1)(R_1^2+I_1^2)$$
$$+(\tfrac{1}{2})(1+2\rho)(1-\rho)[(R_0R_1)^2+(I_0I_1)^2]$$
$$+(\tfrac{1}{2})(1-2\rho)(1+\rho)[(I_0R_1)^2+(R_0I_1)^2]+2\rho R_0I_0R_1I_1$$
$$-[1-\rho^2]^{\frac{1}{2}}\{R_0I_0(R_1^2-I_1^2)+R_1I_1(I_0^2-R_0^2)\}.$$

R_1, I_1, R_0 and I_0 are defined as

$$R_1 = ReH_1^{(1)}$$
$$I_1 = ImH_1^{(1)}$$
$$R_0 = ReH_0^{(1)}$$
$$I_0 = ImH_0^{(1)}$$

where $H_0^{(1)}$ and $H_1^{(1)}$ are the Hankel function of the first kind of order zero and one, respectively.

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Абстракт—Исходя из последних экспериментальных результатов известно, что тонкостенные сферические оболочки, подверженные одномерному внешнему давлению, деформируются в осесимметрическом направлении, в начале процесса выпучивания. Затем исследуется процесс выпучивания и послекритическое поведение полных сферических оболочек. Предполагаются при этом, что как неумышленные, произвольные, науальные отклонения от точной формы, так м последующие упругие деформации являются симметрическими по отношению радиуса оболочки. Далее предполагается, что оболочку можно разложитъ на крышку, которая характеризуется болъшими деформациами и на осталъную частъ, в которой перемещения являются так малыми, что можно удовлетворителъно использоватъ при их определении линейную теорию. Приводится численно к минимуму полная потенциалъная энергия системы. В процессе минималзации условия на границе между крышкой и осталъной частъю точно выполняются. Получаются полные и непреривные кривые послекритического выпучивания для некоторых случаев. Определяется максималъное значение параметра давления в достаточно широким диапазоне геометрического параметра, с целъю получения резулътатов касающихся практического поведения сферических оболочек, загруженных внешным давлением. Даются удовлетворителъные сравнения с предыдущими теоретическими и эксперименталъными работами.